Combinatorial 3-manifolds with a transitive cyclic automorphism group

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Abstract

In this article we substantially extend the classification of combinatorial 3-manifolds with transitive cyclic automorphism group up to 22 vertices. Moreover, several combinatorial criteria are given to decide, whether a cyclic combinatorial d-manifold can be generalized to an infinite family of such complexes together with a construction principle in the case that such a family exist. In addition, a new infinite series of cyclic neighborly combinatorial lens spaces of infinitely many distinct topological types is presented.

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1 Introduction

An *abstract simplicial complex* C can be seen as a combinatorial structure consisting of tuples of elements of $\mathbb{Z}_n$ where the elements of $\mathbb{Z}_n$ are referred to as the vertices of the complex (cf. [17]). The *automorphism group* $\mathrm{Aut}(C)$ of C is the group of all permutations $\sigma \in S_n$ which do not change C as a whole. If $\mathrm{Aut}(C)$ acts transitively on the vertices, C is called a *transitive* simplicial complex. The most basic types of transitive simplicial complexes are the ones which are invariant under the cyclic $\mathbb{Z}_n$-action $v \mapsto v+1 \mod n$, i. e. all complexes C, such that $\mathbb{Z}_n$ is a subgroup of $\mathrm{Aut}(C)$ where n denotes the number of vertices of C. Such complexes are called *cyclic* simplicial complexes.

Many types of cyclic combinatorial structures have been investigated under several different aspects of combinatorics (see for example [18, Part V] for a work on cyclic Steiner systems in the field of design theory). This article is written in the context of combinatorial topology. Hence, we will concentrate on combinatorial manifolds, a special class of simplicial complexes which are defined as follows: An abstract simplicial complex M is said to be *pure*, if all of its tuples are of length $d+1$, where d is referred to as the *dimension* of M. If, in addition, any vertex link of M, i. e. the boundary of a simplicial neighborhood of a vertex of M, is a triangulated $(d-1)$-sphere endowed with the standard piecewise linear structure, M

is called a *combinatorial d-manifold*. There are several articles about cyclic combinatorial d-manifolds, see [15, 21] for many examples and further references.

One major advantage when dealing with simplicial complexes with large automorphism groups is that the complexes can be described efficiently just by the generators of its automorphism group and a system of orbit representatives of the complex under the group action. In the case of a cyclic automorphism group, the situation is particularly convenient. Since, possibly after a relabeling of the vertices, the whole complex does not change under a vertex-shift of type $v \mapsto v+1 \mod n$, two tuples are in one orbit if and only if the differences modulo n of its vertices are equal. Hence, we can compute a system of orbit representatives by just looking at the differences modulo n of the vertices of all tuples of the complex. This motivates the following definition.

Definition 1.1 (Difference cycle). Let $a_i \in \mathbb{N}$, $0 \leq i \leq d$, $n := \sum_{i=0}^{d} a_i$ and $\mathbb{Z}_n = \langle (0, 1, \ldots, n-1) \rangle$. The simplicial complex

$$(a_0 : \ldots : a_d) := \mathbb{Z}_n \langle 0, a_0, \ldots, \Sigma_{i=0}^{d-1} a_i \rangle$$

is called *difference cycle of dimension d on n vertices* where $G\langle\cdot\rangle$ denotes the G-orbit of $\langle\cdot\rangle$. The number of elements of $(a_0 : \ldots : a_d)$ is referred to as the *length* of the difference cycle. If a complex C is a union of difference cycles of dimension d on n vertices and λ is a unit of $\mathbb{Z}_n$ such that the complex λC (obtained by multiplying all vertex labels modulo n by λ) equals C, then λ is called a *multiplier* of C.

Note that for any unit $\lambda \in \mathbb{Z}_n^{\times}$, the complex λC is combinatorially isomorphic to C. In particular, all $\lambda \in \mathbb{Z}_n^{\times}$ are multipliers of the complex $\bigcup_{\lambda \in \mathbb{Z}_n^{\times}} \lambda C$ by construction. The definition of a difference cycle above is equivalent to the one given in [17].

In the following, we will describe cyclic simplicial complexes and cyclic combinatorial manifolds as a set of difference cycles. In this way, a lot of problems dealing with cyclic combinatorial manifolds can be solved in an elegant way. In particular, they play an important role in most of the proofs presented in this article.

Most calculations presented in this work were done with the help of a computer. In particular, the `GAP`-package `simpcomp` [8, 7, 9] as well as `GAP` [10] itself was used to handle difference cycles, permutation groups and quotients of free groups. In addition, the 3-manifold software `regina` by Burton [5] was used for the recognition of the topological type of some combinatorial 3-manifolds.

2 Classification of cyclic 3-manifolds

Neighborly combinatorial 3-manifolds with dihedral automorphism with up to 19 vertices as well as neighborly combinatorial 3-manifolds with cyclic automorphism group with up to 15 vertices have already been classified by Kühnel and Lassmann in 1985, see [15]. Later, a more general classification of all transitive combinatorial manifolds with up to 13 vertices was presented by Lutz in [21] (which also contains a classification of all transitive combinatorial d-manifolds up to 15 vertices in the cases $d \leq 3$ and $d \geq 9$). More recently, Lutz extended

the classification of transitive combinatorial 2-manifolds up to 21 vertices (cf. [22]) and the classification of transitive combinatorial 3-manifolds up to 17 vertices (cf. [20]). All classifications are based on an algorithm first described in [15]. As of Version 1.3, this classification algorithm is also available within simpcomp allowing us to extend any kind of classification of transitive simplicial complexes without the need for any further programming.

In a series of computer calculations, we computed all cyclic combinatorial 3-manifolds with up to 22 vertices. This led to the following result.

Theorem 2.1 (Classification of cyclic combinatorial 3-manifolds)**.** *There are exactly* 6070 *combinatorial types of (connected) combinatorial* 3*-manifolds with cyclic automorphism group with up to* 22 *vertices. These complexes split up into exactly* 67 *topological types.*

The exact number of complexes, combinatorial types, locally minimal complexes and topological types can be found in Table 1. A list of all topological types of 3*-manifolds together with a particular complex of each type sorted by their model geometries is shown in Table 2 to 9. An overview of all topological types of cyclic combinatorial 3-manifolds sorted by vertex number is listed in Table 10.*

All cyclic manifolds are available within simpcomp by calling the function `SCCyclic3Mfld(n,k)` where `n` is the number of vertices and `k` is the number of a specific cyclic combinatorial 3-manifold. The total number of cyclic `n`-vertex combinatorial 3-manifolds can be obtained using the function `SCNrCyclic3Mfld(n)`.

Proof. The complexes were found using the classification algorithm for transitive combinatorial manifolds integrated to the software package simpcomp.

The topological distinctions of most of the spherical and flat manifolds, as well as the connected sums of $S^2 \times S^1$ and $S^2 \times S^1$ were done via comparison of the simplicial homology groups and the fundamental group of the complexes:

- The manifolds of type $(S^2 \times S^1)^{\#k}$ and $(S^2 \underline{\times} S^1)^{\#k}$ were identified by calculating the fundamental group – the free group on k generators – and applying Kneser's conjecture, proved by Stallings in 1959 (see [31]) together with [11, Theorem 5.2].

- By the elliptization conjecture (stated by Thurston in [33, Chapter 3], recently proved by Perelman, see [25, 27, 26]), the topological type of a spherical 3-manifold distinct from a lens space is already determined by the isomorphism type of its (finite) fundamental group. This allows an identification of all such 3-manifolds using the finite group recognition algorithm of GAP.

- The fundamental group distinguishes all flat 3-manifolds by a theorem of Bieberbach (see [3] and [23, page 4]). On the other hand, all other 3-manifolds with a fundamental group containing $\mathbb{Z}^3$ are known to be the connected sum of a flat 3-manifold with some other 3-manifold (cf. [19]). Hence, all 3-manifolds with the fundamental group of a flat manifold have to be prime (as all flat manifolds are prime and the fundamental group of a 3-manifold M determines the length of a prime decomposition of M, cf.

[31] and [11, Theorem 5.2]) and thus are flat. Altogether, the topological type of a 3-manifold with the fundamental group of a flat manifold is in fact flat and the manifold is determined by its fundamental group. Hence, it can be identified using simpcomp and GAP.

For more information about the spherical case in the classification of 3-manifolds see [32, 24], for more about flat 3-manifolds see [3, 23, 14].

Now let us prove that the following complex

$$C := \{(1:1:1:15),(1:2:5:10),(1:5:2:10),(1:5:10:2),(2:5:2:9),(2:6:4:6),(2:7:2:7),(4:4:4:6)\}$$

is homeomorphic to the lens space $L(5,1)$:

Figure 2.1 shows the *slicing*, i. e. the pre-image of a polyhedral Morse function or *regular simplexwise linear function* (see [12]) as described in [30], of C between the odd labeled vertices and the even labeled vertices. Here, the slicing is a torus. Also, both the span of the odd and the span of the even labeled vertices is a solid torus and hence C is a manifold of Heegaard genus 1. For the 1-homology of the two tori $T_- := \partial(\mathrm{span}(0,2,\ldots,16))$ and $T_+ := \partial(\mathrm{span}(1,3,\ldots,17))$ we choose a basis as follows:

$$\begin{aligned} \alpha_- &:= \langle 0,10,4,14,8,0\rangle \\ \beta_- &:= \langle 0,12,6,0\rangle \end{aligned}$$

and

$$\begin{aligned} \alpha_+ &:= \langle 1,11,5,15,9,1\rangle \\ \beta_+ &:= \langle 1,13,7,1\rangle \end{aligned}$$

such that $H_1(T_\pm) = \langle \alpha_\pm, \beta_\pm\rangle$, $H_1(\mathrm{span}(0,2,\ldots,16)) = \langle \beta_-\rangle$ and $H_1(\mathrm{span}(1,3,\ldots,17)) = \langle \beta_+\rangle$.

Now, we want to express α_- in terms of α_+ and β_+. With the help of the slicing (the thick line in Figure 2.1 denotes a path homologous to α_- in the slicing) we see that α_- can be transported to the path

$$\langle 17,15,7,5,3,13,11,3,1,17,9,7,17\rangle$$

which entirely lies in T_+. This path is homologous to -5 times β_+ and 4 times α_+ and hence the topological type of C must be $L(-5,4) \cong L(5,1)$.

In the following, we will prove that the complex

$$\begin{aligned} D \quad := \{ \quad & (1:1:1:19),(1:2:5:14),(1:7:12:2),(2:5:2:13),(2:7:2:11), \\ & (2:8:4:8),(2:9:2:9),(2:12:3:5),(4:6:4:8),(4:6:6:6)\} \end{aligned}$$

is homeomorphic to the lens space $L(7,1)$:

Figure 2.2 shows the slicing of D between the odd labeled vertices and the even labeled vertices which is a torus. Also, both the span of the odd and the span of the even labeled vertices is a solid torus and hence D is a manifold of Heegaard genus 1. For the 1-homology

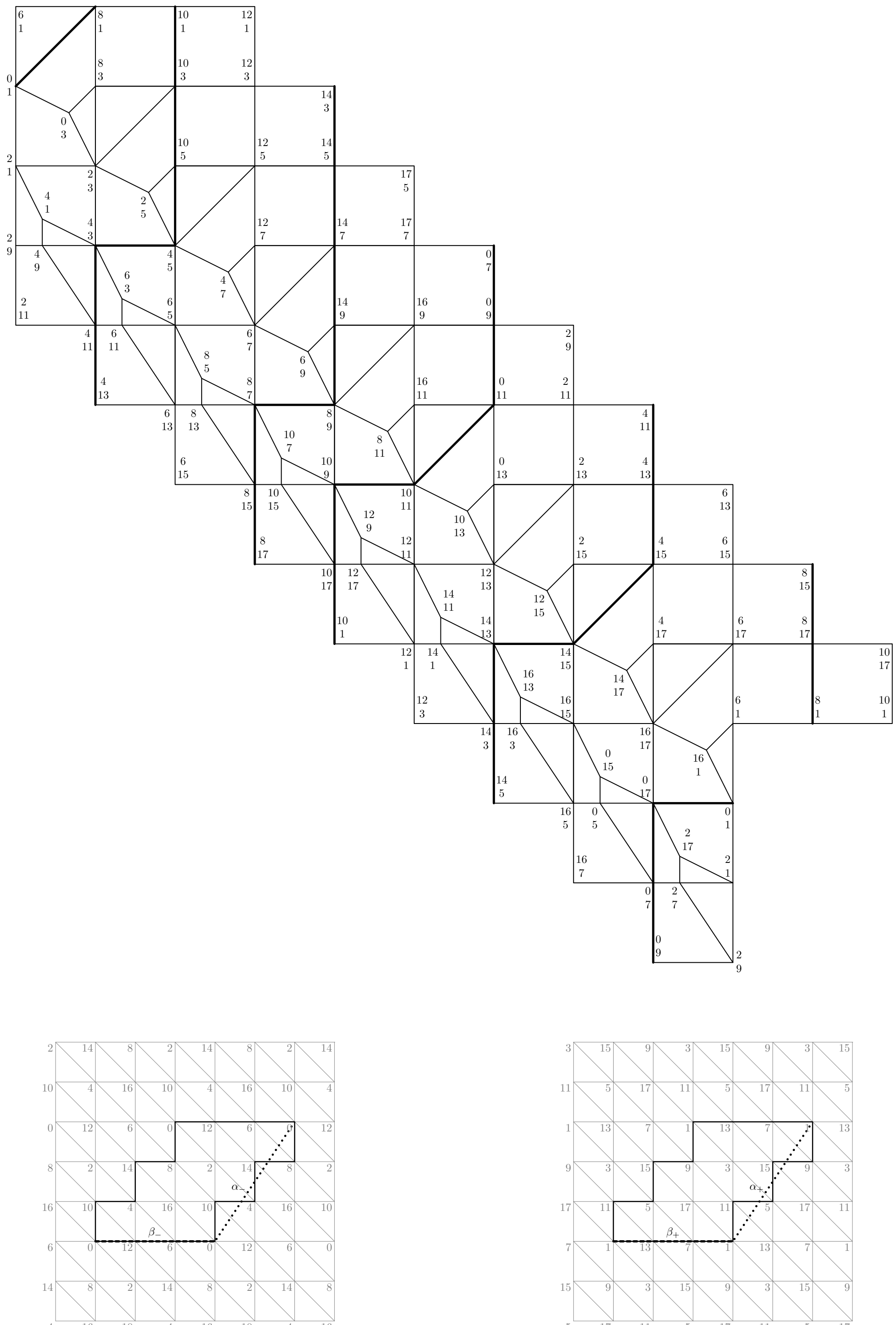

Figure 2.1: Slicing of C between the odd labeled and the even labeled vertices together with the boundary of the two solid tori spanned by the even and by the odd vertices.

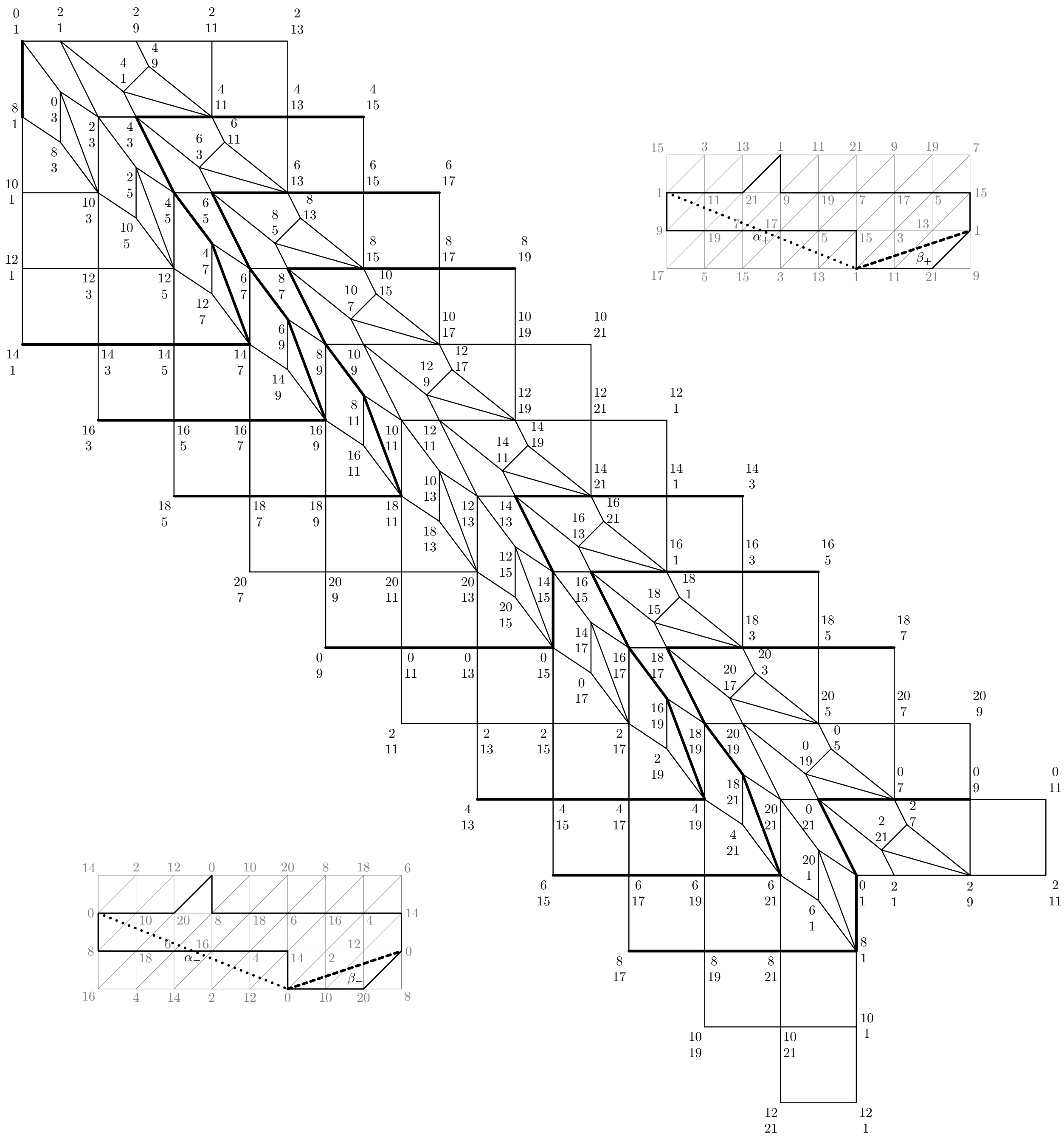

Figure 2.2: Slicing of D between the odd labeled and the even labeled vertices together with the boundary of the two solid tori spanned by the even and by the odd vertices.

of the two tori $T_- := \partial(\mathrm{span}(0,2,\dots,16))$ and $T_+ := \partial(\mathrm{span}(1,3,\dots,17))$ we choose a basis as follows:

$$\begin{aligned}\alpha_- &:= \langle 0,8,18,6,16,4,14,0\rangle\\ \beta_- &:= \langle 0,2,4,0\rangle\end{aligned}$$

and

$$\begin{aligned}\alpha_+ &:= \langle 1,9,19,7,17,5,15,1\rangle\\ \beta_+ &:= \langle 1,3,5,1\rangle\end{aligned}$$

such that $H_1(T_\pm) = \langle \alpha_\pm, \beta_\pm\rangle$, $H_1(\mathrm{span}(0,2,\dots,16)) = \langle\beta_-\rangle$ and $H_1(\mathrm{span}(1,3,\dots,17)) = \langle\beta_+\rangle$.

Once again, we want to express α_- in terms of α_+ and β_+. With the help of the slicing (the thick line in Figure 2.2 denotes a path homologous to α_- in the slicing) we see that α_- can be transported to the path

$$\langle 21,19,17,15,7,5,3,17,15,13,5,3,1,15,13,11,3,1,21,13,11,9,7,21\rangle$$

which entirely lies in T_+. This path is homologous to -7 times β_+ and -1 times α_+ and hence the topological type of D must be $L(-7,-1) \cong L(7,1)$.

For the identification of the exact topological type of the lens spaces

$$\begin{aligned}L_0 &:= \{ && (1:1:1:11),(1:2:4:7),(1:4:2:7),(1:4:7:2),(2:4:4:4),(2:5:2:5)\}\\ L_1 &:= \{ && (1:1:1:15),(1:2:4:11),(1:4:2:11),(1:4:11:2),\\ & && (2:4:8:4),(2:5:2:9),(2:7:2:7),(4:4:4:6)\}\\ L_2 &:= \{ && (1:1:1:19),(1:2:4:15),(1:4:2:15),(1:4:15:2),(2:4:12:4),\\ & && (2:5:2:13),(2:7:2:11),(2:9:2:9),(4:4:4:10),(4:6:4:8)\}\end{aligned}$$

see Theorem 4.1.

All complexes with less than 18 vertices have already been described in literature. See the indicated sources in Table 2, 3, 4 and 8.

The remaining topological types of cyclic 3-manifolds were identified using the 3-manifold software `regina` by Burton [5]. Using `regina`, we also checked that no other types of lens spaces occur in the classification. The notation for the Seifert fibered spaces as well as the graph manifolds is following the one `regina` is using which in turn is based on work by Burton [6] and Orlik [24, pg. 88].

To make sure that none of the Seifert fibered spaces or graph manifolds equal any other topological type of combinatorial 3-manifold previously described in the classification, we additionally computed the Turaev-Viro invariant of the manifolds (see [34]) whenever necessary. See the documentation of `regina` or one of the indicated sources for more information.

In addition, we found a homology 3-sphere which we were not able to identify. It is called HS and is listed in Table 9. $\square$

Table 1: The classification of cyclic combinatorial 3-manifolds with up to 22 vertices

n	# complexes	# cd* compl.	# lm* compl.	# cd lm* compl.	# top. types
5	1	1	1	1	1
6	1	1	0	0	1
7	3	1	0	0	1
8	3	2	0	0	1
9	6	2	3	1	2
10	19	8	0	0	3
11	40	6	0	0	2
12	56	20	0	0	4
13	135	15	0	0	2
14	258	50	0	0	4
15	217	34	1	1	5
16	742	107	12	2	8
17	1272	89	24	2	7
18	1818	319	24	4	15
19	4797	279	63	4	6
20	7670	1008	66	9	20
21	11931	1038	198	18	22
22	30550	3090	230	23	40

* cd = combinatorially distinct, lm = locally minimal

It is interesting to see that some of the homological types of the complexes do not occur for certain integers. Especially, if n is a prime number, the number of homologically distinct complexes seems to be limited. In particular, we believe the following to be true.

Conjecture 2.2. *Let M be a combinatorial 3-manifold with transitive cyclic automorphism group homeomorphic to the total space of the orientable sphere bundle over the circle $S^2 \times S^1$. Then M has an even number of vertices.*

3 Infinite series of combinatorial manifolds

It has always been interesting to see, how cyclic combinatorial manifolds or other highly symmetric complexes can be generalized to a whole family of objects sharing this property. See for example the infinite series of the so-called Altshuler tori with dihedral automorphism group [1, Theorem 4], a family of several infinite series of combinatorial manifolds by Kühnel and Lassmann in [17], a neighborly infinite series of the 3-dimensional Klein bottle in [15] and a neighborly infinite series of the 3-torus in [4].

Table 2: Cyclic combinatorial 3-manifolds of spherical type.

n	top. type	π_1	TV$(7,1)^{***}$	difference cycles of smallest complex*	source
5	S^3	1	0.053787171163	{(1:1:1:2)}	$\partial\Delta^4$
14	$L(3,1)$	$\mathbb{Z}_3$	0.174645847708	{(1:1:1:11), (1:2:4:7), (1:4:2:7), (1:4:7:2), (2:4:4:4), (2:5:2:5)}	[15, Complex 3_{14}], Thm. 4.1
15	$\mathbb{R}P^3$	$\mathbb{Z}_2$		{(1:1:1:12), (1:2:3:9), (1:5:7:2), (2:3:3:7), (3:4:3:5), (3:4:4:4)}	[15, Complex 2_{15}]
15	$P_2 = S^3/Q_8$	Q_8		{(1:1:1:12), (1:2:4:8), (1:6:6:2), (2:4:3:6), (3:4:4:4)}	[15, Complex 8_{15}]
16	$S^3/\mathrm{SL}(2,3)$	SL(2, 3)		{(1:1:3:11), (1:1:4:10), (1:3:2:10), (2:3:8:3), (2:4:6:4), (3:5:3:5)}	[20, Complex ${}^3 16^1_{31}$]
17	Σ^3	SL(2, 5)	0.717779809966	{(1:1:1:14), (1:2:4:10), (1:6:8:2), (2:3:4:8), (2:3:6:6), (2:4:5:6), (4:4:4:5)}	[20, Complex ${}^3 17^1_{21}$]
18	$L(8,3)$	$\mathbb{Z}_8$		{(1:1:1:15), (1:2:4:11), (1:4:2:11), (1:4:11:2), (2:4:8:4), (2:5:2:9), (2:7:2:7), (4:4:4:6)}	Thm. 4.1
18	$L(5,1)$	$\mathbb{Z}_5$		{(1:1:1:15), (1:2:5:10), (1:5:2:10), (1:5:10:2), (2:5:2:9), (2:6:4:6), (2:7:2:7), (4:4:4:6)}	
20	$P_7 = S^3/Q_{28}$	$\mathbb{Z}_7 \rtimes \mathbb{Z}_4$		{(1:1:1:17), (1:2:15:2), (2:3:12:3), (3:4:6:7), (3:4:7:6), (3:5:3:9), (3:6:3:8), (3:6:4:7), (4:6:4:6)}	
22	$P_8 = S^3/Q_{32}$	Q_{32}		{(1:1:1:19), (1:2:1:18), (1:3:15:3), (3:4:3:12), (3:5:6:8), (3:5:8:6), (3:6:3:10), (3:6:5:8), (3:7:3:9), (5:6:5:6)}	
22	$P_4 = S^3/Q_{16}$	Q_{16}		{(1:1:1:19), (1:2:3:16), (1:5:7:9), (1:12:7:2), (2:3:3:14), (2:6:7:7), (3:4:8:7), (3:4:10:5), (4:4:4:10)}	
22	$L(15,4)$	$\mathbb{Z}_{15}$		{(1:1:1:19), (1:2:4:15), (1:4:2:15), (1:4:15:2), (2:4:12:4), (2:5:2:13), (2:7:2:11), (2:9:2:9), (4:4:4:10), (4:6:4:8)}	Thm. 4.1
22	$L(7,1)$	$\mathbb{Z}_7$		{(1:1:1:19), (1:2:5:14), (1:7:12:2), (2:5:2:13), (2:7:2:11), (2:8:4:8), (2:9:2:9), (2:12:3:5), (4:6:4:8), (4:6:6:6)}	

Table 3: Cyclic combinatorial 3-manifolds of type $S^2 \times \mathbb{R}$.

n	top. type	H_*	TV$(7,1)^{***}$	difference cycles of smallest complex*	source
9	$S^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$		{(1:1:2:5), (1:1:5:2), (1:2:1:5)}	[2, Complex N^9_{51}]
10	$S^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	1	{(1:1:2:6), (1:1:6:2), (1:2:1:6)}	[17, Complex $M^3_2(10)$]
17	$\mathbb{R}P^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$		{(1:1:1:14), (1:2:12:2), (2:3:9:3), (3:4:4:6), (3:4:6:4), (3:5:3:6), (3:6:4:4)}	[15, Complex IV_{17}], [20, Complex ${}^3 17^2_{13}$]

Table 4: Cyclic combinatorial 3-manifolds of flat type.

n	top. type	H_*	TV$(7,1)^{***}$	difference cycles of smallest complex*	source
15	$\mathbb{T}^3$	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^3, \mathbb{Z})$		{(1:2:4:8), (1:2:8:4), (1:4:2:8), (1:4:8:2), (1:8:2:4), (1:8:4:2)}	[15, Complex III_{15}]
16	$\mathfrak{B}_2$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	6	{(1:1:3:11), (1:1:4:10), (1:3:2:10), (2:3:4:7), (2:4:7:3), (2:7:3:4)}	[21, p. 89], [20, Complex ${}^3 16^{55}_{10}$]
18	$\mathbb{K}^2 \times S^1$	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$		{(1:1:3:13), (1:1:6:10), (1:3:1:13), (1:6:8:3), (1:7:6:4), (2:3:7:6), (2:6:4:6)}	
18	$\mathfrak{B}_4$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_4, \mathbb{Z}_2, 0)$		{(1:1:3:13), (1:1:13:3), (1:3:1:13), (2:3:6:7), (2:6:2:8), (2:6:7:3), (2:7:2:7), (2:7:3:6)}	
20	$\mathfrak{G}_2$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z})$		{(1:1:3:15), (1:1:6:12), (1:3:1:15), (1:6:10:3), (1:7:2:10), (1:9:6:4), (2:3:7:8), (2:6:5:7), (4:6:4:6)}	
21	$\mathfrak{G}_3$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}, \mathbb{Z})$		{(1:1:1:18), (1:2:1:17), (1:3:5:12), (1:5:3:12), (1:5:6:9), (1:11:6:3), (3:5:8:5), (4:5:6:6)}	

Table 5: Cyclic combinatorial 3-manifolds of Nil type.

n	top. type	H_*	difference cycles of smallest complex*
18	SFS$[\mathbb{T}^2 : (1,1)]$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$	{(1:2:2:13), (1:2:12:3), (1:4:2:11), (1:6:1:10), (1:7:1:9), (1:8:1:8), (1:11:3:3), (2:2:11:3)}
20	SFS$[\mathbb{K}^2/n2 : (1,5)]$	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_4, \mathbb{Z}, \mathbb{Z})$	{(1:1:1:17), (1:2:4:13), (1:6:8:5), (1:8:6:5), (1:8:9:2), (2:4:5:9), (3:4:4:9), (4:4:5:7)}
21	SFS$[S^2 : (3,2)(3,2)(3,-1)]$	$(\mathbb{Z}, \mathbb{Z}_3 \oplus \mathbb{Z}_9, 0, \mathbb{Z})$	{(1:1:1:18), (1:2:1:17), (1:3:6:11), (1:6:3:11), (1:6:11:3), (3:5:3:10), (3:5:8:5), (3:6:6:6), (3:7:3:8)}
21	SFS$[\mathbb{T}^2 : (1,7)]$	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_7, \mathbb{Z}^2, \mathbb{Z})$	{(1:2:4:14), (1:2:5:13), (1:6:5:9), (1:7:9:4), (1:11:3:6), (1:14:2:4), (3:4:3:11), (3:5:9:4), (3:6:5:7)}

Table 6: Cyclic combinatorial 3-manifolds of type $SL(2, \mathbb{R})$.

n	top. type	H_*	TV(7, 1)***	difference cycles of smallest complex*
19	$\Sigma(2,3,7)$	$(\mathbb{Z}, 0, 0, \mathbb{Z})$	0.881772448769	{(1:1:1:16), (1:2:6:10), (1:8:8:2), (2:6:3:8), (3:6:4:6), (4:5:4:6), (4:5:5:5)}
20	SFS[S^2 : (3, 1)(3, 1)(4, −3)]	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	0.838638486511	{(1:1:5:13), (1:1:6:12), (1:5:2:12), (2:5:2:11), (2:6:6:6), (2:7:4:7), (3:4:3:10), (3:4:9:4), (3:7:3:7)}
21	SFS[S^2 : (2, 1)(2, 1)(2, 1)(3, −5)]	$(\mathbb{Z}, \mathbb{Z}_2 \oplus \mathbb{Z}_2, 0, \mathbb{Z})$		{(1:1:1:18), (1:2:7:11), (1:9:9:2), (2:7:3:9), (3:7:4:7), (4:6:4:7), (4:6:5:6), (5:5:5:6)}
21	SFS[S^2 : (5, 1)(5, 1)(5, −4)]	$(\mathbb{Z}, \mathbb{Z}_5 \oplus \mathbb{Z}_{10}, 0, \mathbb{Z})$		{(1:1:3:16), (1:1:4:15), (1:3:10:7), (1:5:8:7), (2:3:6:10), (2:4:10:5), (2:6:3:10), (2:6:8:5), (3:6:3:9)}
21	SFS[S^2 : (4, 1)(4, 1)(4, −3)]	$(\mathbb{Z}, \mathbb{Z}_4 \oplus \mathbb{Z}_4, 0, \mathbb{Z})$		{(1:1:3:16), (1:1:4:15), (1:3:10:7), (1:5:8:7), (2:3:6:10), (2:4:10:5), (2:6:3:10), (2:6:8:5), (3:6:6:6)}
22	SFS[S^2 : (4, 1)(5, 2)(5, −3)]	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$		{(1:1:1:19), (1:2:17:2), (2:3:4:13), (2:7:10:3), (3:4:8:7), (3:5:4:10), (3:5:7:7), (4:6:4:8), (4:6:6:6)}
22	SFS[S^2 : (3, 1)(3, 1)(9, −7)]	$(\mathbb{Z}, \mathbb{Z}_3 \oplus \mathbb{Z}_3, 0, \mathbb{Z})$	0.107574342326	{(1:1:1:19), (1:2:17:2), (2:3:14:3), (3:5:9:5), (4:5:6:7), (4:5:8:5), (4:6:6:6), (4:6:7:5), (4:7:4:7), (4:7:5:6)}
22	SFS[S^2 : (2, 1)(2, 1)(3, 1)(3, −2)]	$(\mathbb{Z}, \mathbb{Z}_{24}, 0, \mathbb{Z})$		{(1:1:3:17), (1:1:4:16), (1:3:2:16), (2:3:7:10), (2:4:2:14), (2:6:8:6), (2:7:2:11), (2:7:10:3), (2:9:2:9), (2:10:3:7)}
22	SFS[S^2 : (3, 2)(4, 1)(4, −3)]	$(\mathbb{Z}, \mathbb{Z}_8, 0, \mathbb{Z})$		{(1:1:3:17), (1:1:4:16), (1:3:2:16), (2:3:14:3), (2:4:12:4), (3:5:3:11), (3:8:3:8), (4:6:6:6)}
22	SFS[S^2 : (3, 1)(3, 1)(5, −3)]	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	0.543133962258	{(1:1:5:15), (1:1:6:14), (1:5:2:14), (2:5:10:5), (2:6:8:6), (3:4:3:12), (3:4:11:4), (3:7:5:7), (4:7:4:7)}

Table 7: Cyclic combinatorial 3-manifolds of type $\mathbb{H}^2 \times \mathbb{R}$.

n	top. type	H_*	TV(7, 1)***	difference cycles of smallest complex*
18	SFS[$\mathbb{R}P^2$: (2, 1)(2, 1)(2, 1)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)$		{(1:1:1:15), (1:2:5:10), (1:4:3:10), (1:4:11:2), (3:4:5:6), (3:5:6:4), (3:6:3:6), (3:6:4:5)}
19	SFS[$\mathbb{R}P^2$: (2, 1)(3, 1)]	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$		{(1:1:1:16), (1:2:5:11), (1:4:3:11), (1:4:12:2), (3:4:6:6), (3:5:3:8), (3:6:4:6), (3:6:6:4)}
20	SFS[$\mathbb{R}P^2$: (3, 1)(3, 2)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_2, 0)$		{(1:1:1:17), (1:2:5:12), (1:5:2:12), (1:5:12:2), (2:5:4:9), (2:6:6:6), (2:9:4:5), (4:5:4:7)}
21	$(\mathbb{R}P^2)^{\#3} \times S^1$	$(\mathbb{Z}, \mathbb{Z}^3 \oplus \mathbb{Z}_2, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$		{(1:1:1:18), (1:2:1:17), (1:3:6:11), (1:6:3:11), (1:6:11:3), (3:5:6:7), (3:5:8:5), (3:6:7:5), (3:7:5:6)}
21	SFS[$\mathbb{R}P^2$: (3, 1)(3, 1)(3, 2)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6, \mathbb{Z}_2, 0)$		{(1:1:1:18), (1:2:5:13), (1:4:3:13), (1:4:14:2), (3:4:3:11), (3:5:6:7), (3:6:6:6), (3:6:7:5), (3:7:5:6)}
21	SFS[$\mathbb{R}P^2$: (3, 1)(3, 1)(3, 1)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_2, 0)$		{(1:1:1:18), (1:2:6:12), (1:4:4:12), (1:4:14:2), (2:5:4:10), (2:6:3:10), (3:4:10:4), (3:6:6:6), (3:10:4:4)}
21	SFS[$\mathbb{K}^2$: (2, 1)]	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	12	{(1:1:3:16), (1:1:13:6), (1:3:11:6), (1:4:9:7), (2:3:6:10), (2:6:4:9), (2:6:10:3), (2:9:7:3), (2:10:3:6)}
21	SFS[$\mathbb{K}^2$: (2, 1)(2, 1)(2, 1)]	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$		{(1:2:3:15), (1:2:13:5), (1:5:2:13), (1:7:4:9), (1:11:4:5), (2:3:3:13), (2:6:4:9), (2:10:4:5), (4:6:4:7)}
22	SFS[$\mathbb{R}P^2$: (3, 1)(4, 3)]	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$		{(1:1:1:19), (1:2:4:15), (1:6:13:2), (2:4:3:13), (3:4:11:4), (3:5:5:9), (3:5:9:5), (3:6:3:10), (3:9:5:5), (4:7:4:7)}
22	SFS[D:(3, 1)(3, 1)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_3, \mathbb{Z}_2, 0)$		{(1:1:1:19), (1:2:10:9), (1:5:7:9), (1:5:14:2), (2:6:8:6), (2:10:4:6), (3:5:10:4), (3:10:5:4), (4:6:7:5)}
22	SFS[$\mathbb{R}P^2$: (2, 1)(5, 1)]	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}_2, 0)$		{(1:1:1:19), (1:2:17:2), (2:3:4:13), (2:7:10:3), (3:4:5:10), (4:5:6:7), (4:6:5:7), (4:6:7:5), (5:6:5:6)}
22	SFS[$\mathbb{K}^2$: (3, 1)]	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	10	{(1:1:3:17), (1:1:4:16), (1:3:2:16), (2:3:14:3), (2:4:9:7), (2:7:6:7), (2:7:9:4), (3:5:3:11), (3:8:3:8)}
22	SFS[S^2 : (2, 1)(2, 1)(3, 1)(3, −4)]	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	4	{(1:1:3:17), (1:1:6:14), (1:3:1:17), (1:6:2:13), (1:7:10:4), (1:8:10:3), (2:3:7:10), (2:6:8:6), (2:10:3:7)}
22	SFS[$\mathbb{R}P^2$: (5, 2)(5, 3)]	$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_5, \mathbb{Z}_2, 0)$		{(1:1:5:15), (1:1:15:5), (1:5:1:15), (2:3:2:15), (2:3:8:9), (2:8:4:8), (2:8:9:3), (2:9:2:9), (2:9:3:8), (4:4:4:10)}
22	SFS[$(\mathbb{R}P^2)^{\#3}$: (1, 1)]	$(\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^2 \oplus \mathbb{Z}_2, 0)$		{(1:1:5:15), (1:1:15:5), (1:5:1:15), (2:5:3:12), (2:8:4:8), (2:9:2:9), (2:9:3:8), (2:11:4:5), (3:8:4:7), (3:10:4:5)}
22	SFS[$\mathbb{K}^2$: (2, 1)(2, 1)]	$(\mathbb{Z}, \mathbb{Z}^2 \oplus \mathbb{Z}_4, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$		{(1:2:4:15), (1:2:13:6), (1:4:2:15), (1:4:8:9), (1:12:3:6), (2:4:8:8), (2:12:3:5), (2:13:3:4), (3:5:8:6)}

Table 8: Cyclic combinatorial 3-manifolds which are connected sums.

n	top. type	H_*	difference cycles of smallest complex*	source
12	$(S^2 \times S^1)^{\#2}$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$	{(1:2:3:6), (1:2:4:5), (1:5:1:5), (2:2:2:6), (2:3:3:4)}	[15, Complex 5_{12}]
16	$(S^2 \times S^1)^{\#5}$	$(\mathbb{Z}, \mathbb{Z}^5, \mathbb{Z}^5, \mathbb{Z})$	{(1:2:5:8), (1:2:6:7), (1:3:4:8), (1:3:5:7), (2:5:3:6), (2:6:2:6), (3:4:4:5)}	[20, Complex ${}^{3}16^{1}_{41}$]
18	$(S^2 \times S^1)^{\#7}$	$(\mathbb{Z}, \mathbb{Z}^7, \mathbb{Z}^7, \mathbb{Z})$	{(1:1:7:9), (1:1:8:8), (1:7:2:8), (2:3:4:9), (2:3:6:7), (3:3:3:9), (3:4:5:6), (4:5:4:5)}	
18	$(S^2 \times S^1)^{\#7}$	$(\mathbb{Z}, \mathbb{Z}^7, \mathbb{Z}^6 \oplus \mathbb{Z}_2, 0)$	{(1:1:7:9), (1:1:9:7), (1:7:1:9), (2:3:4:9), (2:3:6:7), (3:3:3:9), (3:4:5:6), (4:5:4:5)}	
20	$(S^2 \times S^1)^{\#6}$	$(\mathbb{Z}, \mathbb{Z}^6, \mathbb{Z}^6, \mathbb{Z})$	{(1:1:3:15), (1:1:4:14), (1:3:5:11), (1:5:5:9), (1:8:2:9), (2:3:7:8), (2:4:5:9), (3:5:5:7), (3:7:3:7)}	

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n	top. type	H_*	difference cycles of smallest complex	source
20	$(S^2 \times S^1)^{\#6}$	$(\mathbb{Z}, \mathbb{Z}^6, \mathbb{Z}^5 \oplus \mathbb{Z}_2, 0)$	{(1:1:3:15), (1:1:8:10), (1:3:7:9), (1:4:6:9), (1:8:5:6), (1:9:4:6), (2:3:5:10), (3:5:5:7), (3:7:3:7)}	
20	$(S^2 \times S^1)^{\#4}$	$(\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}^4, \mathbb{Z})$	{(1:2:2:15), (1:2:4:13), (1:4:5:10), (1:6:4:9), (1:9:1:9), (2:2:4:12), (2:6:9:3), (3:4:4:9), (4:5:5:6)}	
20	$(S^2 \underline{\times} S^1)^{\#9}$	$(\mathbb{Z}, \mathbb{Z}^9, \mathbb{Z}^8 \oplus \mathbb{Z}_2, 0)$	{(1:2:7:10), (1:2:8:9), (1:4:5:10), (1:4:11:4), (1:10:5:4), (2:6:2:10), (2:6:6:6), (2:7:3:8), (3:7:3:7)}	
21	$(S^2 \times S^1)^{\#12}$	$(\mathbb{Z}, \mathbb{Z}^{12}, \mathbb{Z}^{12}, \mathbb{Z})$	{(1:2:4:14), (1:2:11:7), (1:6:3:11), (1:9:4:7), (2:4:7:8), (3:3:3:12), (3:4:5:9), (3:6:7:5), (3:9:4:5)}	
22	$(S^2 \underline{\times} S^1)^{\#12}$	$(\mathbb{Z}, \mathbb{Z}^{12}, \mathbb{Z}^{11} \oplus \mathbb{Z}_2, 0)$	{(1:1:9:11), (1:1:10:10), (1:9:2:10), (2:3:6:11), (2:3:8:9), (3:4:4:11), (3:4:11:4), (3:6:5:8), (3:11:4:4), (5:6:5:6)}	

Table 9: Cyclic combinatorial 3-dimensional graph manifolds** and a 22-vertex homology sphere of unknown topological type.

n	top. type	H_*	TV(7,1)***	difference cycles of smallest complex*
20	$\mathrm{SFS}[D:(3,1)(3,1)] \cup_m \mathrm{SFS}[D:(3,1)(3,1)]$, $m = \begin{pmatrix} -4 & 5 \\ -3 & 4 \end{pmatrix}$	$(\mathbb{Z}, \mathbb{Z}_3 \oplus \mathbb{Z}_3, 0, \mathbb{Z})$	0.0750935889735	{(1:1:3:15), (1:1:4:14), (1:3:4:12), (1:5:2:12), (2:3:6:9), (2:4:9:5), (2:9:3:6), (3:4:4:9)}
22	HS	$(\mathbb{Z}, 0, 0, \mathbb{Z})$	0.0213064178104	{(1:1:1:19), (1:2:4:15), (1:4:8:9), (1:4:15:2), (1:6:6:9), (2:4:10:6), (2:5:6:9), (2:9:2:9), (2:9:5:6), (4:4:4:10)}
22	$\mathrm{SFS}[D:(2,1)(2,1)] \cup_m \mathrm{SFS}[D:(2,1)(3,1)]$, $m = \begin{pmatrix} -5 & 11 \\ -4 & 9 \end{pmatrix}$	$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z})$	1.87111923986	{(1:1:1:19), (1:2:5:14), (1:7:12:2), (2:4:5:11), (2:4:11:5), (2:5:4:11), (2:8:2:10), (2:12:3:5), (4:5:4:9), (5:6:5:6)}
22	$\mathrm{SFS}[D:(2,1)(3,1)] \cup_m \mathrm{SFS}[D:(3,1)(3,1)]$, $m = \begin{pmatrix} -8 & 11 \\ -5 & 7 \end{pmatrix}$	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	1.55495813209	{(1:1:1:19), (1:2:5:14), (1:4:3:14), (1:4:15:2), (3:4:6:9), (3:5:3:11), (3:6:4:9), (3:6:9:4), (3:8:3:8), (4:6:6:6)}
22	$\mathrm{SFS}[A:(2,1)(2,1)]/m$, $m = \begin{pmatrix} 1 & -11 \\ 1 & -10 \end{pmatrix}$	$(\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}_2, 0)$	23.3297487925	{(1:1:4:16), (1:1:11:9), (1:4:8:9), (1:5:6:10), (2:4:10:6), (2:9:2:9), (2:9:5:6), (3:4:3:12), (3:4:8:7), (3:7:8:4)}

* The smallest complex is the lexicographically (with respect to the difference cycles) minimal complex of all complexes of a given topological type with the smallest number of vertices.

** Graph manifolds consist of Seifert fibered spaces with toroidal boundary components, glued together along homeomorphisms of the boundary components given by an element of the mapping class group of the torus $m \in \mathrm{SL}(2, \mathbb{Z})$. D denotes a disc and A an annulus hence an object with two boundary components.

*** The symbol TV(7,1) denotes the Turaev-Viro invariant (see [34]) with parameters `r = 7` and `whichRoot = 1` as indicated in the documentation of `regina`.

Table 10: Topological types of cyclic combinatorial 3-manifolds.

homology groups	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
S^3	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
$S^3/\mathrm{SL}(2,3)$												×						×
Σ^3													×					×
$P_2 = S^3/Q_8$											×			×			×	
$P_4 = S^3/Q_{16}$																		×
$P_7 = S^3/Q_{28}$																×		
$P_8 = S^3/Q_{32}$																		×
$L(3,1)$										×		×				×	×	×
$L(5,1)$														×				×
$L(7,1)$																		×
$L(8,3)$														×		×		×
$L(15,4)$																		×
$S^2 \times S^1$						×		×		×		×		×		×		×
$S^2 \underline{\times} S^1$					×	×	×	×	×	×	×	×	×	×	×	×	×	×
$\mathbb{R}P^2 \times S^1$													×		×	×	×	×
$\mathbb{R}P^3$											×		×	×			×	×
$\mathbb{T}^3$											×	×	×	×	×	×	×	×
$\mathfrak{B}_2$												×	×	×	×	×	×	×
$\mathfrak{B}_4$														×		×		×
$\mathfrak{G}_2$																×		

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Table 10 – continued from previous page

homology groups	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\mathfrak{G}_3$																	×	
$\mathbb{K}^2 x S^1$														×		×	×	×
$\Sigma(2,3,7)$															×			
$(S^2 \times S^1)^{\#2}$								×										
$(S^2 \times S^1)^{\#4}$																×		
$(S^2 \times S^1)^{\#5}$												×						
$(S^2 \times S^1)^{\#6}$																×		
$(S^2 \tilde{\times} S^1)^{\#6}$																×		
$(S^2 \times S^1)^{\#7}$														×				
$(S^2 \tilde{\times} S^1)^{\#7}$														×				
$(S^2 \tilde{\times} S^1)^{\#9}$																×		
$(S^2 \times S^1)^{\#12}$																	×	×
$(S^2 \tilde{\times} S^1)^{\#12}$																		×
$\mathrm{SFS}[\mathbb{T}^2 : (1,1)]$														×				
$\mathrm{SFS}[\mathbb{T}^2 : (1,7)]$																	×	
$(\mathbb{R}P^2)^{\#3} \times S^1$																	×	
$\mathrm{SFS}[D_:(3,1)(3,1)]$																		×
$\mathrm{SFS}[\mathbb{K}^2/n2 : (1,5)]$																×		
$\mathrm{SFS}[\mathbb{K}^2 : (2,1)(2,1)(2,1)]$																	×	
$\mathrm{SFS}[\mathbb{K}^2 : (2,1)(2,1)]$																		×
$\mathrm{SFS}[\mathbb{K}^2 : (2,1)]$																	×	×
$\mathrm{SFS}[\mathbb{K}^2 : (3,1)]$																		×
$\mathrm{SFS}[(\mathbb{R}P^2)^{\#3} : (1,1)]$																		×
$\mathrm{SFS}[\mathbb{R}P^2 : (2,1)(2,1)(2,1)]$														×			×	
$\mathrm{SFS}[\mathbb{R}P^2 : (2,1)(3,1)]$																		×
$\mathrm{SFS}[\mathbb{R}P^2 : (2,1)(5,1)]$																		×
$\mathrm{SFS}[\mathbb{R}P^2 : (3,1)(3,1)(3,1)]$																	×	
$\mathrm{SFS}[\mathbb{R}P^2 : (3,1)(3,1)(3,2)]$																	×	
$\mathrm{SFS}[\mathbb{R}P^2 : (3,1)(3,2)]$																×		×
$\mathrm{SFS}[\mathbb{R}P^2 : (3,1)(4,3)]$																		×
$\mathrm{SFS}[\mathbb{R}P^2 : (5,2)(5,3)]$																		×
$\mathrm{SFS}[S^2 : (2,1)(2,1)(2,1)(3,-5)]$																	×	
$\mathrm{SFS}[S^2 : (2,1)(2,1)(3,1)(3,-2)]$																		×
$\mathrm{SFS}[S^2 : (2,1)(2,1)(3,1)(3,-4)]$																		×
$\mathrm{SFS}[S^2 : (3,1)(3,1)(4,-3)]$																×		
$\mathrm{SFS}[S^2 : (3,1)(3,1)(5,-3)]$																		×
$\mathrm{SFS}[S^2 : (3,1)(3,1)(9,-7)]$																		×
$\mathrm{SFS}[S^2 : (3,2)(3,2)(3,-1)]$																	×	
$\mathrm{SFS}[S^2 : (3,2)(4,1)(4,-3)]$																		×
$\mathrm{SFS}[S^2 : (4,1)(4,1)(4,-3)]$																	×	
$\mathrm{SFS}[S^2 : (4,1)(5,2)(5,-3)]$																		×
$\mathrm{SFS}[S^2 : (5,1)(5,1)(5,-4)]$																	×	
$\mathrm{SFS}[D : (3,1)(3,1)] \cup_m \mathrm{SFS}[D : (3,1)(3,1)], m = \begin{pmatrix} -4 & 5 \\ -3 & 4 \end{pmatrix}$																×		
HS																		×
$\mathrm{SFS}[D : (2,1)(2,1)] \cup_m \mathrm{SFS}[D : (2,1)(3,1)], m = \begin{pmatrix} -5 & 11 \\ -4 & 9 \end{pmatrix}$																		×
$\mathrm{SFS}[D : (2,1)(3,1)] \cup_m \mathrm{SFS}[D : (3,1)(3,1)], m = \begin{pmatrix} -8 & 11 \\ -5 & 7 \end{pmatrix}$																		×
$\mathrm{SFS}[A : (2,1)(2,1)]/m, m = \begin{pmatrix} 1 & -11 \\ 1 & -10 \end{pmatrix}$																		×

In the case of combinatorial complexes with cyclic automorphism group, a generalization of a given complex to an infinite series of such triangulations with increasing number of vertices seems somewhat natural. One way to see this uses slicings of combinatorial 3-manifolds as described in [29, Section 4.2]. The idea is to generalize a slicing of a combinatorial 3-manifold extending the cyclic symmetry. More generally, in the case of a cyclic combinatorial 3-manifold represented by a set of difference cycles, there is a simple combinatorial condition whether a given triangulation can be generalized to an infinite family of cyclic complexes or not.

Theorem 3.1. *Let $M = \{d_1, \dots, d_m\}$ be a combinatorial 3-manifold with n vertices, represented by m difference cycles $d_i = (a_i^0 : \dots : a_i^3)$, $1 \le i \le m$. Without loss of generality let us assume that $a_i^3 \ge a_i^j$ for all $1 \le i \le m$, $0 \le j \le 2$.*

Then the complex $M_k = \{d_{1,k}, \dots, d_{m,k}\}$ with $d_{i,k} = (a_i^0 : \dots : a_i^3 + k)$, $1 \le i \le m$, is a combinatorial manifold for all $k \ge 0$ if and only if $a_i^3 > a_i^0 + \dots + a_i^2$ for all $1 \le i \le m$.

In order to prove Theorem 3.1 let us first take a look at a few lemma.

Lemma 3.2. *Let $(a_0 : \dots : a_d)$ be a difference cycle of dimension d on n vertices and $1 \le k \le d+1$ the smallest integer such that $k \mid (d+1)$ and $a_i = a_{i+k}$, $0 \le i \le d-k$. Then $(a_0 : \dots : a_d)$ is of length $\sum_{i=0}^{k-1} a_i = \frac{nk}{d+1}$.*

Proof. We set $m := \frac{nk}{d+1}$ and compute

$$\begin{aligned}\left\langle 0+m, a_0+m, \dots, (\Sigma_{i=0}^{d-1} a_i) + m \right\rangle &= \left\langle \Sigma_{i=0}^{k-1} a_i, \Sigma_{i=0}^{k} a_i, \dots, \Sigma_{i=0}^{d-1} a_i, 0, a_1, \dots, \Sigma_{i=0}^{k-2} a_i \right\rangle \\ &= \left\langle 0, a_0, \dots, \Sigma_{i=0}^{d-1} a_i \right\rangle\end{aligned}$$

(all entries are computed modulo n). Hence, for the length l of $(a_0 : \dots : a_d)$ we have $l \le \frac{nk}{d+1}$ and since k is minimal with $k \mid (d+1)$ and $a_i = a_{i+k}$, the upper bound is attained. □

Lemma 3.3. *Let M_k, $k \ge 0$, be an infinite series of cyclic combinatorial 3-manifolds with $n+k$ vertices represented by the union of m difference cycles of full length, that is, the length of the difference cycles equals the number of vertices $n+k$ of the complex. Then we have for the f-vector of the series*

$$f\left(\mathrm{lk}_{M_0}(0)\right)) = f\left(\mathrm{lk}_{M_k}(0)\right)) = (2m+2, 6m, 4m)$$

for all $k \ge 0$. In particular, the number of vertices of $\mathrm{lk}_{M_k}(0)$ does not depend on the value of k.

Proof. Since M_k is the union of m difference cycles of full length, we have for the number of tetrahedra $f_3(M_k) = m(n+k)$ for all $k \ge 0$. Furthermore, as M_k is cyclic, all vertices are contained in the same number of tetrahedra which has 4 vertices. By the fact that any facet of $\mathrm{lk}_{M_k}(0)$ corresponds to a facet in M_k containing 0 it follows that for the number of triangles of the link $f_2(\mathrm{lk}_{M_k}(0)) = \frac{4m(n+k)}{n+k} = 4m$ holds, which is independent of k. Since for all $k \ge 0$ M_k is a combinatorial 2-sphere, all edges of lk_{M_k} lie in exactly two triangles, hence $f_1(\mathrm{lk}_{M_k}(0)) = 6m$. Finally, the Euler characteristic of the 2-sphere is 2, and by the Euler-Poincaré formula we have $f_0(\mathrm{lk}_{M_k}(0)) = 2m+2$. □

Let us now come to the proof of Theorem 3.1.

Proof. Now let $M = \{d_1, \dots, d_m\}$ be a combinatorial 3-manifold with n vertices, represented by m difference cycles $d_i = (a_i^0 : \dots : a_i^3)$, $1 \le i \le m$, such that $a_i^3 > a_i^0 + \dots + a_i^2$ for all $1 \le i \le m$.

For the link of vertex 0 in M we then have:

$$\mathrm{lk}_M(0) = \bigcup_{i=1}^{m} \bigcup_{j=-1}^{2} \langle -\sum_{k=0}^{j} a_i^k, \dots, -a_i^j, a_i^{j+1}, \dots \sum_{k=j+1}^{2} a_i^k \rangle \tag{3.1}$$

which has to be a triangulated 2-sphere, as M is a combinatorial 3-manifold. Since $a_i^3 > \frac{n}{2} > a_i^0 + \dots + a_i^2$ for all $1 \le i \le m$, the vertices $v_j \in \{0, \dots n-1\}$ of $\mathrm{lk}_M(0)$ can be mapped to the vertices of $\mathrm{lk}_{M_k}(0)$, $k \ge 0$, as follows:

$$v_j \mapsto \begin{cases} v_j & \text{if } v_j < \frac{n}{2} \\ v_j + k & \text{if } v_j \ge \frac{n}{2}. \end{cases}$$

This yields a combinatorial isomorphism between $\mathrm{lk}_M(0)$ and $\mathrm{lk}_{M_k}(0)$. Since M and M_k are cyclic, all vertex links are isomorphic. Altogether it follows that M_k is a combinatorial manifold for all $k \ge 0$.

This part of the proof can be generalized to combinatorial d-manifolds, d arbitrary, see Theorem 3.7.

Conversely, let $M = \{d_1, \dots, d_m\}$ be a combinatorial 3-manifold with n vertices, represented by m difference cycles $d_i = (a_i^0 : \dots : a_i^3)$, $1 \le i \le m$, such that $M_k = \{d_{1,k}, \dots, d_{m,k}\}$ with $d_{i,k} = (a_i^0 : \dots : a_i^3 + k)$, $1 \le i \le m$, is a combinatorial manifold for all $k \ge 0$. Now, there exist a $\tilde{k} \ge 0$ such that $a_i^3 + \tilde{k} = a_i^0 + \dots + a_i^2$ for one difference cycle d_i and $a_j^3 + \tilde{k} \ge a_j^0 + \dots + a_j^2$ for all other $1 \le j \le m$. Since $a_j^3 + \tilde{k} \ge a_j^0 + \dots + a_j^2$ and $a_j^l > 0$ for all $1 \le j \le m$, $0 \le l \le 3$, it follows by Lemma 3.2 that all difference cycles of $M_{\tilde{k}}$ and $M_{\tilde{k}+1}$ have full length. By Lemma 3.3 it now follows that the links of vertex 0 in $M_{\tilde{k}}$ and $M_{\tilde{k}+1}$ have the same f-vector. On the other hand, since $a_i^3 + \tilde{k} = a_i^0 + \dots + a_i^2$ but $a_j^3 + \tilde{k} + 1 > a_j^0 + \dots + a_j^2$ for all $1 \le j \le m$, we can see by looking at the vertices of $\mathrm{lk}_{M_{\tilde{k}}}(0)$ that $\mathrm{lk}_{M_{\tilde{k}+1}}(0)$ has to have strictly more vertices than the link of vertex 0 in $M_{\tilde{k}}$. This is a contradiction to Lemma 3.3. □

Remark 3.4. Theorem 3.1 shows, how a single cyclic combinatorial 3-manifold can be extended to an infinite number of combinatorial 3-manifolds by adding an arbitrary positive integer to the largest entry in every difference cycle. More generally, we will talk about *infinite series of cyclic combinatorial d-manifolds* whenever the infinite family of complexes is constructed by adding multiples of a positive integer k to certain entries of the difference cycles of a combinatorial d-manifold M of arbitrary dimension d. In contrast to that, in Section 4 we will look at an infinite series with an increasing number of difference cycles. Hence, infinite series of combinatorial d-manifolds can be defined in various ways. As a consequence, in every context attention has to be payed what exactly is meant by an infinite series of combinatorial manifolds.

In the following, we will require an infinite series of cyclic combinatorial manifolds to start with the smallest complex that is a combinatorial manifold, that is, the complex M_{-1} must not be a combinatorial manifold.

Corollary 3.5. *Let M_k, $k \geq 0$, be an infinite series of cyclic combinatorial 3-manifolds such that M_{-1} is not a combinatorial manifold, then M_0 has an odd number of vertices.*

Proof. This follows immediately by the fact, that $\Delta_j := a_j^d - a_j^0 - \ldots - a_j^{d-1} > 0$ for all $1 \leq j \leq m$ in M_0. If the mimimum over all Δ_j, $1 \leq j \leq m$, is greater than 1, M_{-1} is a combinatorial 3-manifold by Theorem 3.1 and M_0 is not the smallest member of that infinite series. Hence, $\Delta_i = 1$ for some $1 \leq i \leq m$ and $n = 2a_i^d + 1$. □

Another direct consequence from the classification and Theorem 3.1 is the following result.

Corollary 3.6. *There are exactly 396 combinatorially distinct dense infinite series of combinatorial 3-manifolds starting with a triangulation with less than 23 vertices.*

So far, we just considered infinite series of cyclic combinatorial manifolds that have members for all integers $n \geq n_0$ for n_0 sufficiently large. However, the notion of an infinite series of combinatorial manifolds as described in Remark 3.4 is more general. In fact, there are other (weaker) formulations of infinite series of cyclic combinatorial d-manifolds: In the following, we will call a series N_k *of order* l, $l \in \mathbb{N}$, if there exist an integer $n_0 \in \mathbb{N}$ such that there are triangulations with $n = n_0 + k \cdot l$ vertices in N_k for all $k \geq 0$. The case $l = 1$ contains all other cases. It coincides with the previously described series and will be referred to as a *dense* series.

There is an analogue to the first half of Theorem 3.1 for infinite series of combinatorial d-manifolds of order l, $1 < l \leq d$, which can be formulated as follows.

Theorem 3.7. *Let $N = \{d_1, \ldots, d_m\}$ be a combinatorial d-manifold with n vertices, represented by m difference cycles $d_i = (a_i^0 : \ldots : a_i^d)$, $1 \leq i \leq m$.*

Let $N_k = \{d_{1,k}, \ldots, d_{m,k}\}$ be a simplicial complex with $n + lk$ vertices, $l \in \mathbb{N}$ fixed, $k \geq 0$, defined by $d_{i,k} = (a_i^0 + l_i^0 k : \ldots : a_i^d + l_i^d k)$, $1 \leq i \leq m$, where for each $1 \leq i \leq m$ we have $\sum_{j=0}^d l_i^j = l$, $l_i^j \geq 0$.

Then N_k is a combinatorial d-manifold for all $k \geq 0$ if

$$\frac{(l_i^j + 1)n}{l+1} > a_i^j > \frac{l_i^j n}{l+1},$$

holds for all $0 \leq i \leq d$, $0 \leq j \leq d$.

Proof. The proof is completely analogue to the one of the first part of Theorem 3.1. Here, too, we look at a relabeling of the vertices of the link $\mathrm{lk}_N(0)$ in order to transform it to $\mathrm{lk}_{N_k}(0)$.

The relabeling is given by

$$v_j \mapsto v_j + \left\lfloor \frac{(d+1)v_j}{n} \right\rfloor k.$$

The first half of Theorem 3.1 corresponds to the case $d = 3$ and $l = 1$. □

Theorem 3.7 defines series of order l, $1 \le l \le d$, by a purely combinatorial criterion. Since all dense series contain series of order l, the following characterisation of higher order series is interesting.

Lemma 3.8. *Let $N_k = (d_{1,k}, \ldots, d_{m,k})$, $k \ge 0$, be an infinite series of combinatorial d-manifolds of order l, $1 \le l \le d$, with $n + lk$ vertices given as in Theorem 3.7 by non-negative integers l_i^j, $1 \le i \le m$, $0 \le j \le d$, $\sum_{j=0}^{d} l_i^j = l$. Then the following holds.*

If l is a unit in $\mathbb{Z}_n$, all but finitely many members of N_k, $k \ge 0$, are contained in a dense series.

Proof. Let $a_{i,k}^j$ be the j-th entry of the i-th difference cycle of N_k. By multiplying N_k by l we get $lN_k = \{(la_{1,k}^0 : \ldots : la_{1,k}^d), \ldots, (la_{m,k}^0 : \ldots : la_{m,k}^d)\}$. Hence, we have $la_{i,k}^j = la_i^j + ll_i^j k = la_i^j - l_i^j n$ which is independent of k and by adding $n + lk$ to each of the a_i^d, $1 \le i \le m$, we get $\sum_{j=0}^{d} la_{1,k}^j = n + lk$.

Now, if $k = 0$, N_0 has n vertices, and l is a unit in $\mathbb{Z}_n$, the multiplied complex lN_0 is a combinatorial manifold and, thus, all differences of lN_0 are non-zero. Since, in lN_k, only $a_{i,k}^d$ depends on k it follows, that for $k \ge k_0$ sufficiently large we can i) rearrange all differences such that all differences are greater than zero and ii) Theorem 3.7 in the case $l = 1$ can be applied. Hence, all N_k, $k \ge k_0$, are contained in an infinite dense series of combinatorial d-manifolds. □

Corollary 3.9. *Let N_k, $k \ge 0$, be an infinite series of cyclic combinatorial d-manifolds of order 2 which is not contained in a dense series. Then the number of vertices of N_0 has to be even.*

Proof. This follows immediately since 2 is a unit in $\mathbb{Z}_n$ for all $n \equiv 1(2)$. □

Since Theorem 3.7 is valid for arbitrary dimensions, an extended classification of cyclic combinatorial manifolds of higher dimensions would certainly lead to further interesting results. However, this is work in progress.

4 An infinite series of neighborly lens spaces of varying topological types

The infinite series described in Section 3 as well as all other infinite series of transitive combinatorial 3-manifolds described in literature contain only a few topological types of 3-manifolds: There are series known with members of type $S^2 \times S^1$ or $S^2 \underline{\times} S^1$ (see [15] or [29, Section 4.2]), S^3 (the the boundary of the cyclic 4-polytopes), $\mathbb{T}^3$ or $\mathfrak{B}_2$ (see [4],

[17] or series number 17 from Corollary 3.6, `SCSeriesK(17,k)` in `simpcomp`) or the series with number 30, 42 and 356 from Corollary 3.6 (`SCSeriesK(30,k)`, `SCSeriesK(42,k)` and `SCSeriesK(356,k)` in `simpcomp`) which contain a few more combinatorial 3-manifolds and up to three distinct topological types per series.

However, in dimension 2 several infinite series of transitive combinatorial surfaces with changing topological type exist. There is a series of neighborly orientable surfaces of genus $\frac{1}{6}\binom{12s+4}{2}$ with $12s+7$ vertices (cf. [28, Fig. 2.15] and [13, Example 2.7]) starting with the 7-vertex Möbius torus. In addition, a lot of further series of transitive combinatorial surfaces with similar properties can be found in [22] by Lutz.

There are infinite series containing manifolds of increasing dimension and thus containing infinitely many topologically distinct members – but the manifolds are mostly of the same class: The boundary of the d-simplex $\partial\Delta^d$, the boundary of the cross polytopes $\partial\beta^d$ and the boundary of the cyclic polytopes $\partial\delta C(d+1,n)$ are prominent examples of infinite series of d-spheres, there is an infinite series of d-tori $\mathbb{T}^d$ in [16] and there is a series of sphere bundles, spheres and tori M_k^d in [17]). However, none of the above series contains a lot of topologically distinct members of a fixed dimension.

Thus, neighborly series of combinatorial 3-manifolds which additionally have members of many different topological types would be interesting to investigate. Unfortunately, due to the higher complexity such series are hard to find. However, using the large amount of complexes from the classification described in Section 2, the following infinite series of topologically distinct lens spaces could be constructed.

Theorem 4.1. *The complex*

$$\begin{aligned} L_k \;:=\; & \{ (1:1:1:11+4k), (1:2:4:7+4k), (1:4:2:7+4k), (1:4:7+4k:2) \} \\ & \bigcup_{i=0}^{k} \{ (2:5+2i:2:5+4k-2i), (4:2+2i:4:4+4k-2i) \} \end{aligned} \tag{4.1}$$

is a combinatorial 3*-manifold with* $n = 14+4k$*,* $k \geq 0$*, vertices. It is homeomorphic to the lens space* $L((k+2)^2-1, k+2)$*.*

Proof. Obviously, L_k has $n = 14+4k$ vertices. By looking at Figure 4.1 we can verify that the link $\mathrm{lk}_{L_k}(0)$ of vertex 0 in L_k is a triangulated 2-sphere. Hence, as L_k has transitive symmetry it follows immediately that L_k is in fact a combinatorial 3-manifold for all $k \geq 0$. Furthermore, we can see that $\mathrm{lk}_{L_k}(0)$ has $13+4k$ vertices and thus L_k is 2-neighborly. To determine the exact topological type of L_k we will proceed as follows:

1. for all $k \geq 0$, determine a Heegaard splitting $T_k^- \cup_{S_k} T_k^+$ of L_k of genus 1,
2. draw the center torus S_k of the splitting as a slicing (see Figure 4.2),
3. choose a base $H_1(\partial T_k^-) = \langle \alpha_k^-, \beta_k^- \rangle$ of the 1-homology of the boundary of the lower solid torus T_k^- such that $H_1(T_k^-) = \langle \beta_k^- \rangle$,
4. do the same for the upper solid torus T_k^+ such that $H_1(\partial T_k^+) = \langle \alpha_k^+, \beta_k^+ \rangle$ and $H_1(T_k^+) = \langle \beta_k^+ \rangle$,

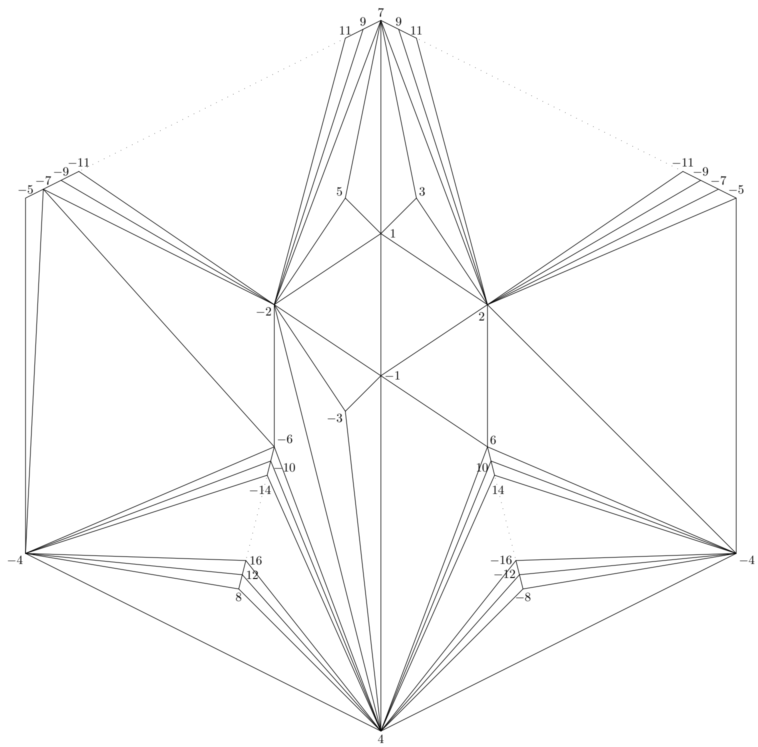

Figure 4.1: Link of vertex 0 of L_k – a triangulated 2-sphere with $13+4k$ vertices.

5. determine the homological type of α_k^- in $H_1(\partial T_k^+)$ – by construction this will be a torus knot which will determine the topological type of L_k.

1. For all $k \geq 0$, the span of the even labeled vertices $T_k^- := \mathrm{span}(\{0,2,\ldots,n-1\})$ as well as the span of the odd labeled vertices $T_k^+ := \mathrm{span}(\{1,3,\ldots,n\})$ (which is combinatorially isomorphic to T_k^- by the cyclic symmetry) form a solid torus and hence the slicing between the odd and the even vertices $S_k := S_{(\{0,2,\ldots\},\{1,3,\ldots\})}(L_k)$ is a torus.

To see this note that T_k^- together with T_k^+ are exactly the difference cycles

$$T_k^- \cup T_k^+ = \bigcup_{i=0}^{k} \{\, (4 : 2+2i : 4 : 4+4k-2i) \,\} \subset L_k.$$

Since the gcd of 4, $2+2i$ and $4+4k-2i$, $0 \leq i \leq k$, is 2 for all $k \geq 0$, T_k^- and T_k^+ are disjoint but connected and we have

$$T_k^- \cong T_k^+ \cong \bigcup_{i=0}^{k} \{\, (2 : 1+i : 2 : 2+2k-i) \,\} =: T_k.$$

For $k=0$ we have $T_0 = \{(1:1:1:4)\} \cong B^2 \times S^1$. Now let $k \geq 1$. T_k consists of $k+1$ difference cycles and we will note $\delta_i := (2 : 1+i : 2 : 2+2k-i)$. δ_i shares two triangles per tetrahedron with δ_{2+i}, $0 \leq i \leq k-2$, δ_{k-1} shares two triangles per tetrahedron with δ_k, $k \geq 1$, δ_1 shares two triangle per tetrahedron with itself and δ_0 shares two triangles per tetrahedron with ∂T_k and

hence contains the complete boundary of T_k. Altogether, we have the following collapsing sheme of T_k:

$$\begin{tikzcd}[column sep=small]
\partial T_k \arrow[r, <->] & \delta_0 \arrow[r, phantom, "\circlearrowright"] \arrow[rr, bend left, <->] & \delta_1 \arrow[rr, bend right, <->] & \delta_2 \arrow[rr, bend left, <->] & \delta_3 \arrow[rr, bend right] & \delta_4 & \cdots \arrow[rr, bend right] & \delta_{k-4} \arrow[rr, bend left, <->] & \delta_{k-3} \arrow[rr, bend right, <->] & \delta_{k-2} \arrow[rr, bend left, <->] & \delta_{k-1} \arrow[r, bend right, <->] & \delta_k
\end{tikzcd}$$

Thus, T_k collapses onto $\delta_1 = (2:2:2:1+2k)$ and since the modulus of δ_1 is odd we have $\delta_1 \cong (1:1:1:4+2k) \cong B^2 \times S^1$. As a direct consequence, $T_k^- \cup_{S_k} T_k^+$ defines a Heegaard splitting of L_k of genus 1 and L_k is homeomorphic to the 3-sphere, $S^2 \times S^1$ or a lens space $L(p,q)$.

2. The center piece of the Heegaard splitting $S_k := S_{(\{0,2,\ldots\},\{1,3,\ldots\})}(L_k)$ is shown in Figure 4.2. It is interesting to see that apart from T_k^- and T_k^+ the difference cycles $(1:2:4:7+4k)$ and $(1:4:2:7+4k)$ are the only ones which do not contain two odd and two even labels per tetrahedron and thus are the only ones which are not sliced by S_k in a quadrilateral. Hence, S_k consists of only $28+8k$ triangles but $(2+k)(14+4k)+7+2k = 4k^2+24k+35$ quadrilaterals. Its complete f-vector is

$$f(S_k) = (4k^2+28k+49, 8k^2+60k+112, (8k+28)\Delta, (4k^2+24k+35)\Box).$$

3. and **4.** In order to find a suitable basis of $H_1(\partial T_k^-)$ as indicated above, let us first take a look at ∂T_k^- itself which is shown in Figure 4.3. We choose the Basis of $H_1(\partial T_k^-) = \langle \alpha_k^-, \beta_k^- \rangle$ to be

$$\begin{aligned}
\alpha_k^- &= \langle 0,4,8,\ldots,n-6,0 \rangle \\
\beta_k^- &= \langle 0,6,12,18,22,26,\ldots,n-4,0 \rangle
\end{aligned}$$

or in the case that $n < 26$ as indicated in Figure 4.3. By construction, α_k^- is contractible in T_k^- and $H_1(T_k^-) = \langle \beta_K^- \rangle$.

For $H_1(\partial T_k^+) = \langle \alpha_k^+, \beta_k^+ \rangle$ we choose analogously

$$\begin{aligned}
\alpha_k^+ &= \langle 1,5,9,\ldots,n-5,1 \rangle \\
\beta_k^+ &= \langle 1,7,13,19,23,27,\ldots,n-3,1 \rangle
\end{aligned}$$

and hence $H_1(T_k^+) = \langle \beta_K^+ \rangle$.

5. To finish the proof we will express α_k^- in terms of α_k^+ and β_k^+. This is done by a map $\phi : H_1(\partial T_k^-) \to H_1(\partial T_k^+)$ which lifts any path in L_k passing only even labeled vertices (a path in ∂T_k^-) to a homologically equivalent path passing only odd labeled vertices (a path in ∂T_k^+). The image of a path under ϕ can be determined with the help of the slicing S_k. In the case of α_k^- it is the thick line in Figure 4.2 and results in the following path:

$$\begin{aligned}
\phi(\alpha_k^-) = \langle\ & n-7, n-9, n-11, \ldots, 9, 7, 1, n-1, n-3, \\
& n-3, n-5, n-7, \ldots, 13, 11, 5, 3, 1, \\
& 1, n-1, n-3 \ldots, 17, 15, 9, 7, 5, \\
& \ldots \\
& n-13, n-15, n-17, \ldots, 3, 1, n-5, n-7 \quad \rangle.
\end{aligned} \tag{4.2}$$

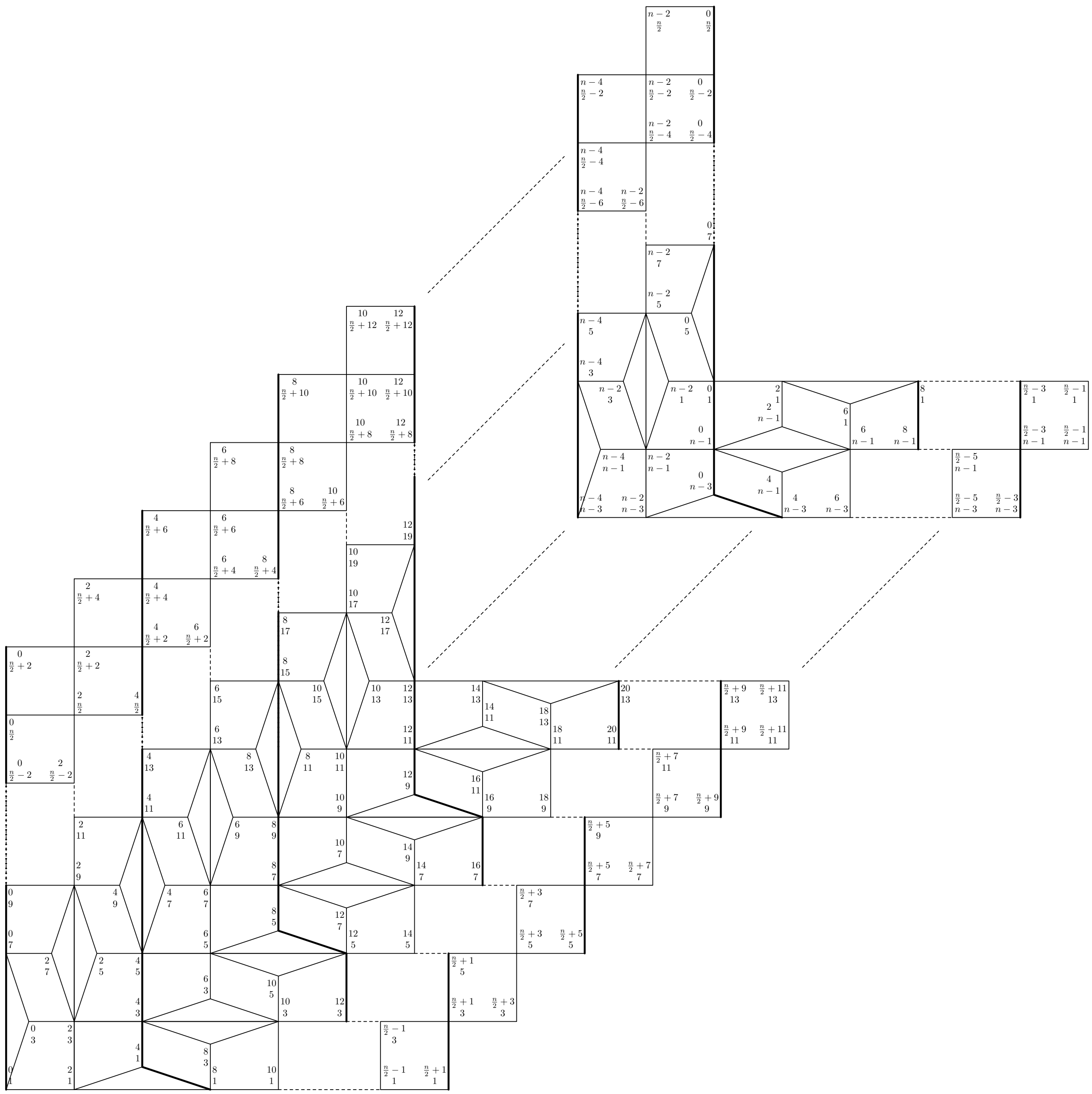

Figure 4.2: Slicing of L_k between the odd labeled and the even labeled vertices – a triangulated torus.

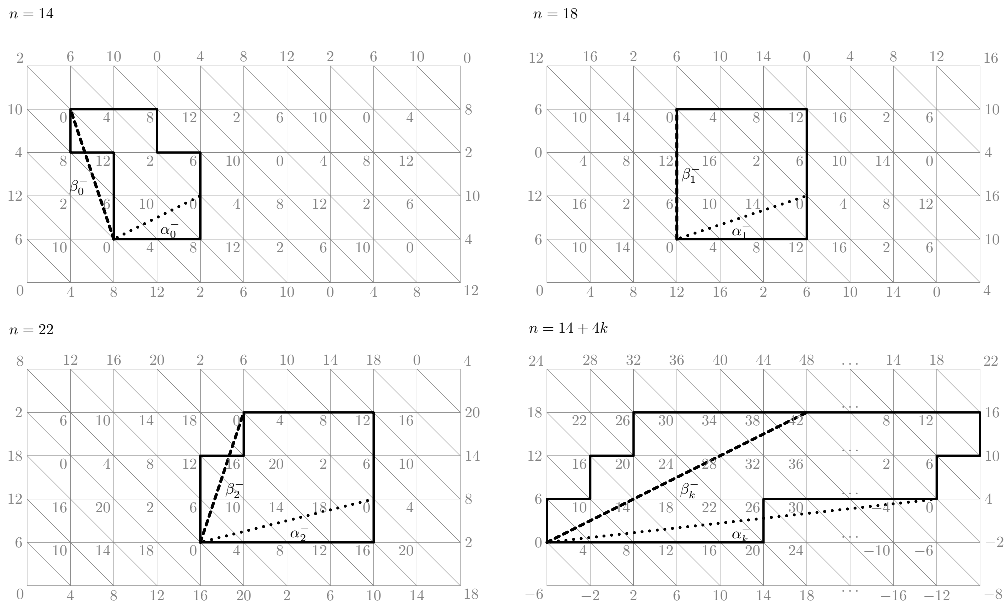

Figure 4.3: Fundamental domain of the boundary of T_k^- together with the basis $\langle \alpha_k^-, \beta_k^- \rangle$ of $H_1(\partial T_k^-)$ for selected values of k and in greater generality.

By taking a closer look to Figure 4.3 we see that all edges of a path of type $\langle s, s-2 \rangle$ in both ∂T_k^- and ∂T_k^+ go from the left upper corner of a square of the grid to the lower right corner ($\searrow$) whereas an edge of type $\langle s, s-6 \rangle$ is simply going down in the grid ($\downarrow$). As $\phi(\alpha_k^-)$ has $(k+2)(2k+2)+2k+1$ segments of type $\searrow$ and $k+3$ segments of type $\downarrow$, $\phi(\alpha_k^-)$ results in the vector $(2k^2+8k+5, 2k^2+9k+8)$ on the integer grid with basis $(\rightarrow, \downarrow)$ (cf. Figure 4.3 where ∂T_k^+ is obtained from ∂T_k^- by the shift $v \mapsto (v+1) \mod n$ of all vertex labels).

On the other hand, we know that α_k^+ corresponds to the vector $(k+2, -1)$ and β_k^+ to $(k-1, -3)$ on the grid for ∂T_k^+ with basis $(\rightarrow, \downarrow)$. Thus, to express $\phi(\alpha_k^-)$ in terms of α_k^+ and β_k^+ we have to solve the following system of equations:

$$\begin{array}{lrcrcl} \text{I.} & (k+2)q & + & (k-1)p & = & 2k^2+8k+5 \\ \text{II.} & -\quad q & - & 3p & = & 2k^2+9k+8 \end{array} \tag{4.3}$$

which results in the solution

$$q = k^2 + 3k + 1; \quad p = -k^2 - 4k - 3$$

and hence

$$\phi(\alpha_k^-) = (k^2+3k+1)\alpha_k^+ + (-k^2-4k-3)\beta_k^+.$$

Furthermore, note that $L(p, q_1) \cong L(p, q_2)$ if and only if $q_1 \equiv \pm q_2^{\pm 1} \mod p$ from which it follows that

$$K_k \cong L((k+2)^2 - 1, k+2).$$

$\square$

The series L_k can be modified into a series of 3-spheres which only differs to L_k in the part which is disjoint to the slicing S_k. Hence, Theorem 4.1 shows that combinatorial surgery of infinitely many essentially different types can be applied in a setting respecting the cyclic symmetry of the underlying combinatorial manifolds. The following corollary, which is a direct implication of Theorem 4.1, summarizes the findings of this section under a more general point of view.

Corollary 4.2. *There are infinitely many topologically distinct combinatorial (prime)* 3-*manifolds with transitive cyclic automorphism group.*

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